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A BORSUK-ULAM TYPE GENERALIZATION OF THE LERAY-SCHAUDER FIXED POINT THEOREM

Anatoliy K. Prykarpatsky[1]
The Department of Nonlinear Mathematical Analysis at the IAPMM of NAS, Lviv, 79060, Ukraine,
The AGH University of Science and Technology, Krakow 30059, Poland
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

A generalization of the classical Leray-Schauder fixed point theorem, based on the infinite-dimensional Borsuk-Ulam type antipode construction, is proposed. Two completely different proofs based on the projection operator approach and on a weak version of the well known Krein-Milman theorem are presented.

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[1]prykanat@cybergal.com; pryk.anat@ua.fm

1. Introduction

The classical Leray-Schauder fixed point theorem and its diverse versions [2, 1, 5, 8, 11, 13, 18, 15] in infinite-dimensional both Banach and Frechet spaces, being nontrivial generalizations of the well known finite-dimensional Brouwer fixed point theorem, have many very important applications [2, 5, 8, 11, 12, 10] in modern applied analysis. In particular, there exist many problems in theories of differential and operator equations [2, 12, 17, 18, 10, 15], which can be uniformly formulated as

$$\hat{a}\ x = f(x), \tag{1.1}$$

where $\hat{a} : E_1 \to E_2$ is some closed surjective linear operator from Banach space E_1 into Banach space E_2, defined on a domain $D(\hat{a}) \subset E_1$, and $f : E_1 \to E_2$ is some, in general, nonlinear continuous mapping, whose domain $D(f) \subseteq D(\hat{a}) \cap S_r(0)$, with $S_r(0) \subset E_1$ being the sphere of radius $r \in \mathbb{R}_+$ centered at zero. Concerning the mapping $f : E_1 \to E_2$ we will assume that it is $\hat{a}$ -compact. This means that the induced mapping $f_{gr} : D_{gr}(\hat{a}) \to E_2$, where $D_{gr}(\hat{a}) \subset E_1 \oplus E_2$ is the extended graph domain endowed with the graph-norm, Lipschitz-projected onto the space E_1 via $j : D_{gr}(\hat{a}) \subset E_1$, and the following equality $f_{gr}(\bar{x}) = f(j(\bar{x}))$ holds for any $\bar{x} \in D_{gr}(\hat{a})$. It is easy to observe also [9] that the mapping $f : E_1 \to E_2$ is $\hat{a}$-compact if and only if it is continuous and for any bounded set $A_2 \subset E_2$ and arbitrary bounded set $A_1 \subset D(f)$ the set $f(A_1 \cap \hat{a}^{-1}(A_2))$ is relatively compact in E_2. The empty set $\varnothing$, by definition, is considered to be compact too.

2. Preliminary constructions

Assume that a continuous mapping $f : E_1 \to E_2$ satisfies the following conditions:

1) the domain $D(f) = D(\hat{a}) \cap S_r(0)$;

2) the mapping $f : D(f) \to E_2$ is $\hat{a}$ - compact;

3) there holds a bounded constant $k_f > 0$, such that $\sup\limits_{y \in S_r(0)} \frac{1}{r} \|f(y)\|_2 = k_f^{-1}$,

where a linear operator $\hat{a} : E_1 \to E_2$ is taken closed and surjective with the domain $D(\hat{a}) \subset E_1$. The domain $D(\hat{a})$, in general, can not be dense in E_1.

Let now $\tilde{E}_1 := E_1 / Ker\ \hat{a}$ and $p_1 : E_1 \to \tilde{E}_1$ be the corresponding projection. The induced mapping $\tilde{a} : \tilde{E}_1 \to E_2$ with the domain $D(\tilde{a}) := p_1(D(\hat{a}))$ is defined as usual, that is for any $\tilde{x} \in D(\tilde{a})$, $\hat{a}(\tilde{x}) := a(p_1(\tilde{x}))$. It is a well know fact [1, 13, 18] that the mapping $\tilde{a} : \tilde{E}_1 \to E_2$ is invertible and its norm is calculated as

$$\left\|\tilde{a}^{-1}\right\| := \sup_{\|y\|_2 = 1} \left\|\tilde{a}^{-1}(y)\right\| = \sup_{\|y\|_2 = 1} \inf_{x \in D(\hat{a})} \left\{\|x\|_1 : a(x) = y\right\}, \tag{2.1}$$

where we denoted by $\|\cdot\|_1$ and $\|\cdot\|_2$ the corresponding norms in spaces E_1 and E_2. The following standard lemma [13, 18] holds.

Lemma 2.1. *The mapping $\tilde{a} : \tilde{E}_1 \to E_2$ is invertible and the norm $\left\|\tilde{a}^{-1}\right\| := k(\hat{a}) < \infty$.*

Proof. We have, by definition (2.1), that the norm $\|\tilde{a}^{-1}\|$ equals

$$k(\hat{a}) = \|\tilde{a}^{-1}\| := \sup_{y\in E_2} \frac{\|\tilde{a}^{-1}(y)\|_{\tilde{E}_1}}{\|y\|_2} = \sup_{y\in E_2} \frac{1}{\|y\|_2} \inf_{x\in D(\hat{a})} \{\|x\|_1 : \hat{a}(x) = y\}. \tag{2.2}$$

Since the linear mapping $\hat{a} : E_1 \to E_2$ is surjective, the mapping $\hat{a}^{-1} : E_2 \to \tilde{E}_1$ is defined on the whole space E_2. Moreover, as the mapping $\hat{a} : E_1 \to E_2$ is a closed operator, the induced inverse operator $\tilde{a}^{-1} : E_2 \to \tilde{E}_1$ is closed [13, 17, 18] too. Thereby, making use of the classical closed graph theorem [1, 12, 13], we conclude that the inverse operator $\tilde{a}^{-1} : E_2 \to \tilde{E}_1$ is bounded, that is norm

$$\|\tilde{a}^{-1}\| := k(\hat{a}) < \infty, \tag{2.3}$$

finishing the proof. □

The next lemma characterizes the multi-valued mapping $\hat{a}^{-1} : E_2 \to E_1$ by means of the constant $k(\hat{a}) < \infty$, defined by (2.3).

Lemma 2.2. *The multi-valued inverse mapping $\hat{a} : E_2 \to E_1$ is Lipschitzian with the Lipschitz constant $k(\hat{a}) < \infty$, that is*

$$\rho_\chi(\hat{a}^{-1}(y_1), \hat{a}^{-1}(y_2)) \le k(\hat{a}) \|y_1 - y_2\|_2 \tag{2.4}$$

for any $y_1, y_2 \in E_2$, where $\rho_\chi : \tilde{E}_1 \times \tilde{E}_1 \to \mathbb{R}_+$ is the standard Hausdorf metrics [1, 13, 18] *in the space E_1.*

Proof. The statement is a simple corollary from formula (2.2) and the Hausdorf metrics definition. □

To describe the solution set of equation (1.1) we need to know a more deeper structure of the mapping $\hat{a} : E_1 \to E_2$ and its multi-valued inverse $\hat{a}^{-1} : E_2 \to E_1$. Namely, we are interested in finding a suitable, in general, nonlinear continuous selection $s : E_2 \to E_1$ [1, 12, 15, 14] of the multi-valued mapping $\hat{a}^{-1} : E_2 \to E_1$, satisfying some additional properties.

The following theorem is crucial for proving the main result obtained below.

Lemma 2.3. *For any constant $k_s > k(\hat{a})$ there exists a continuous odd mapping $s : E_2 \to E_1$, satisfying the following conditions: i) $\hat{a}(s(y)) = y$ for any $y \in E_2$; ii) $\|s(y)\|_1 \le k_s \|y\|_2$, $y \in E_2$.*

Proof. Since the multi-valued mapping $\hat{a}^{-1} : E_2 \to E_1$ is defined on the whole Banach space E_2, one can write down that

$$\hat{a}^{-1}\, y = \bar{x}_y \oplus Ker\ \hat{a} \tag{2.5}$$

for any $y \in E_2$ and some specified elements $\bar{x}_y \in E_1 \backslash Ker\ \hat{a}$, labelled by elements $y \in E_2$. If the composition (2.5) is already specified, we can define a selection $s : E_2 \to E_1$ as follows:

$$s(y) := \frac{1}{2}(\bar{x}_y - \bar{x}_{-y}) \oplus \frac{1}{2}(\bar{c}_y - \bar{c}_{-y}), \tag{2.6}$$

where the elements $\bar{c}_y \in Ker\ \hat{a}$, $y \in E_2$, are chosen arbitrary, but fixed. It is now easy to check that

$$s(-y) = -s(y) \tag{2.7}$$

and

$$\begin{aligned} \hat{a}\ s(y) &= \hat{a}\ (\frac{1}{2}(\bar{x}_y - \bar{x}_{-y}) \oplus \frac{1}{2}(\bar{c}_y - \bar{c}_{-y})) \\ &= \frac{1}{2}\hat{a}\ \bar{x}_y - \frac{1}{2}\hat{a}\ \bar{x}_{-y} = \frac{1}{2}y - \frac{1}{2}(-y) = y \end{aligned} \tag{2.8}$$

for all $y \in E_2$, thereby the mapping (2.6) satisfies the main conditions $i)$ and $ii)$ above. To state the continuity of the mapping (2.6), we will consider below expression (2.2) for the norm $\|\tilde{a}^{-1}\| = k(\hat{a})$ of the linear mapping $\tilde{a}^{-1} : E_2 \to \tilde{E}_1$. We can easily write down the following inequality

$$\begin{aligned} \|s(y)\|_1 &= \left\|\frac{1}{2}(\bar{x}_y - \bar{x}_{-y}) \oplus \frac{1}{2}(\bar{c}_y - \bar{c}_{-y})\right\|_1 \\ &= \frac{1}{2}\|(\bar{x}_y \oplus \bar{c}_y) - (\bar{x}_{-y} \oplus \bar{c}_{-y})\|_1 \\ &\leq \frac{1}{2}(\|(\bar{x}_y \oplus \bar{c}_y)\|_1 + \|(\bar{x}_{-y} \oplus \bar{c}_{-y})\|_1) \\ &\leq \frac{1}{2}k_s\|y\|_2 + \frac{1}{2}k_s\|y\|_2 = k_s\|y\|_2\,, \end{aligned} \tag{2.9}$$

giving rise to the continuity of mapping (2.6), where we have assumed that there exists such a constant $k_s > 0$, that

$$\|(\bar{x}_y \oplus \bar{c}_y)\|_1 \leq k_s\|y\|_2\,, \tag{2.10}$$

for all $y \in E_2$. This constant $k_s > k(\hat{a})$ strongly depends on the choice of elements $\bar{c}_y \in Ker\ \hat{a}$, $y \in E_2$, what one can observe from definition (2.2). Really, owing to the definition of infimum, for any $\varepsilon > 0$ and all $y \in E_2$ there exist elements $\bar{x}_y^{(\varepsilon)} \oplus \bar{c}_y^{(\varepsilon)} \in E_1$, such that

$$k(\hat{a}) \leq \frac{\left\|\bar{x}_y^{(\varepsilon)} \oplus \bar{c}_y^{(\varepsilon)}\right\|_1}{\|y\|_2} < k(\hat{a}) + \varepsilon := k_s. \tag{2.11}$$

Now making now use of formula (2.6), we can construct a selection $s_\varepsilon : E_2 \to E_1$ as follows:

$$s_\varepsilon(y) := \frac{1}{2}(\bar{x}_y^{(\varepsilon)} - \bar{x}_{-y}^{(\varepsilon)}) \oplus \frac{1}{2}(\bar{c}_y^{(\varepsilon)} - \bar{c}_{-y}^{(\varepsilon)}), \tag{2.12}$$

satisfying, owing to inequalities (2.11), the searched for conditions $i)$ and $ii)$:

$$\hat{a}\ s_\varepsilon(y) = y, \qquad \|s_\varepsilon(y)\|_1 \leq k_s\|y\|_2 \tag{2.13}$$

for all $y \in E_2$ and $k_s := k(\hat{a}) + \varepsilon$, $\varepsilon > 0$.
Moreover, the mapping $s_\varepsilon : E_2 \to E_1$ is, by construction, continuous [14, 6, 9] and odd that finishes the proof. □

3. An infinite - dimensional Borsuk-Ulam type generalization of the Leray-Schauder fixed point theorem

Consider now the equation (1.1), where mappings $\hat{a}: E_1 \to E_2$ and $f: E_1 \to E_2$ satisfy the conditions described above. Moreover, we will assume that the selection $s: E_2 \to E_1$, constructed above, and the mapping $f: D(f) \subset E_1 \to E_2$ satisfy additionally the following inequalities:

$$k(\hat{a}) < k_s < k_f \,, \tag{3.1}$$

where, by definition,

$$\sup_{x \in S_r(0)} \frac{1}{r} \|f(x)\| := k_f^{-1} < \infty. \tag{3.2}$$

Then the following main theorem holds.

Theorem 3.1. *Assume that the dimension* $\dim Ker\ \hat{a} \geq 1$, *then equation (1.1) possesses on the sphere* $S_r(0) \subset E_1$ *the nonempty solution set* $\mathcal{N}(\hat{a}, f) \subset E_1$, *whose topological dimension* $\dim \mathcal{N}(\hat{a}, f) \geq \dim Ker\ \hat{a} - 1$.

Proof. Suppose that $\dim Ker\ \hat{a} \geq 1$ and state first that the set $\mathcal{N}(\hat{a}, f)$ is nonempty. Consider a reduced mapping $f_r: D(\hat{a}) \subset E_1 \to E_2$, where

$$f_r(x) := \left\{ \begin{array}{ll} \frac{\|x\|_1}{r} f(\frac{rx}{\|x\|_1}), & \text{if } \ x \neq 0 \\ 0, & \text{if } \ = 0 \end{array} \right\} \tag{3.3}$$

and observe that this mapping is $\hat{a}$ - compact too, if the mapping $f: D(f) \subset E_1 \to E_2$ was taken $\hat{a}$ - compact. Really, for any bounded sets $A_2 \subset E_2$ and $A_1 \subset B_R(0) \cap D(\hat{a})$ the set

$$f_r(A_1 \cap \hat{a}^{-1}(A_2)) \subset \left\{ ty \in E_2 : t \in [0, R/r]\,, y \in f(S_r(0)) \cap \hat{a}^{-1}(A_2) \right\} := F_r \tag{3.4}$$

is relatively compact owing to the $\hat{a}$ - compactness of the mapping $f: D(f) \subset E_1 \to E_2$, where $B_R(0)$ is a ball of radius $R > 0$. Thereby, the closed set $\bar{F}_r \subset E_2$ is compact, or the mapping (3.3) is $\hat{a}$ - compact.

Assume now that a mapping $s: E_2 \to E_1$ satisfies all of the conditions formulated in Theorem 2.3. Take a nonzero element $\bar{c} \in Ker\ \hat{a}$, define the Banach space $E_2^{(+)} := E_2 \oplus \mathbb{R}$ and consider a set of mappings $\varphi_r^{(\varepsilon)}: E_2^{(+)} \to E_2$, where

$$\varphi_r^{(\varepsilon)}(y,t) := \frac{t}{t^2 + \varepsilon^2} f_r(ts(y) + t^2 \bar{c}) \tag{3.5}$$

for all $(y,t) \in E_2^{(+)}$, small enough $\varepsilon \in \mathbb{R} \backslash \{0\}$ and some fixed nontrivial element $\bar{c} \in Ker\ \hat{a}$. It is also evident that

$$\varphi_r^{(\varepsilon)}(y, 0) := 0, \tag{3.6}$$

being well definite for all $\varepsilon \in \mathbb{R} \backslash \{0\}$ and $y \in E_2$, owing to condition 3) imposed above on the mapping $f: D(f) \subset E_1 \to E_2$. The set of mappings (3.5) is, evidently, odd, that is

$$-\varphi_r^{(\varepsilon)}(y,t) = \varphi_r^{(\varepsilon)}(-y,-t) \tag{3.7}$$

for all $(y,t) \in E_2^{(+)}$, $\varepsilon \in \mathbb{R}\backslash\{0\}$ and moreover, it is compact. Really, for any bounded set $A_2^{(+)} := A_2 \oplus \Delta \subset E_2^{(+)}$, where $\Delta \subset \mathbb{R}$ is an arbitrary bounded interval, the set $B_2 := \underset{t\in\Delta}{\cup} B_2^{(t)}$, $B_2^{(t)} := \{s(y) + t\bar{c} \in E_2\}$, is bounded too, and $B_2 \subset \hat{a}^{-1}(A_2)$. Owing to the $\hat{a}$ - compactness of mapping (3.3), one gets that the set

$$\varphi_r^{(\varepsilon)}(A_2^{(+)}) = \underset{t\in\Delta}{\cup} \frac{t}{t^2+\varepsilon^2} f_r(tB_2^{(t)}) \tag{3.8}$$

is relatively compact, since all of the sets $f_r(tB_2^{(t)}) \subset E_2$ are relatively compact for any $t \in \Delta$ and, owing to the condition 3) mentioned above, the set $\varphi_r^{(\varepsilon)}(A_2^{(+)})$ is bounded for any $\varepsilon \in \mathbb{R}\backslash\{0\}$. Thereby, the closed set $\overline{\varphi_r^{(\varepsilon)}(A_2^{(+)})} \subset E_2$ for any $\varepsilon \in \mathbb{R}\backslash\{0\}$, meaning that the mapping (3.5) is compact.

Take now the unit sphere $S_1^{(+)}(0) \subset E_2^{(+)}$ and consider the equation

$$\varphi_r^{(\varepsilon)}(y,t) = y \tag{3.9}$$

for $(y,t) \in S_1^{(+)}(0)$ and $\varepsilon \in \mathbb{R}\backslash\{0\}$ that is

$$\|y\|_2^2 + t^2 = 1. \tag{3.10}$$

We assert that equation (3.9) possesses for any $\varepsilon \in \mathbb{R}\backslash\{0\}$ a solution $(y_\varepsilon, t_\varepsilon) \in S_1^{(+)}(0)$, such that $t_\varepsilon \neq 0$ and

$$\frac{t_\varepsilon}{t_\varepsilon^2+\varepsilon^2} f_r(t_\varepsilon s(y_\varepsilon) + t_\varepsilon^2 \bar{c}) = y_\varepsilon\ , \tag{3.11}$$

where the vector $t_\varepsilon s(y_\varepsilon) + t_\varepsilon^2\bar{c} \in E_2$ is nontrivial (i.e. it is not equal to zero!). This is guaranteed by conditions imposed on the mapping $f : S_r(0) \subset E_1 \to E_2$ and the following Borsuk-Ulam type theorem, generalizing the well known Borsuk-Ulam [1, 15, 18, 8] antipode theorem, proved in [9] and formulated below in a convenient for us form.

Theorem 3.2. *Let $E_2^{(+)}$ and E_2 be Banach spaces, $\hat{b} : E_2^{(+)} \to E_2$ be a linear continuous surjective operator, $S_r^{(+)}(0) \subset E_2^{(+)}$ be a sphere of radius $r > 0$ centered at zero of $E_2^{(+)}$ and $\varphi : S_r^{(+)}(0) \to E_2$ be a compact, in general nonlinear, odd mapping. Then if* $\dim Ker\ \hat{b} \geq 1$, *the equation*

$$\hat{b}\ z = \varphi(z), \tag{3.12}$$

$z \in S_r^{(+)}(0)$, *possesses the nonempty solution set $\mathcal{N}(\hat{b},\varphi) \subset E_2^{(+)}$, whose topological dimension* $\dim \mathcal{N}(\hat{b},\varphi) \geq \dim Ker\ \hat{b} - 1$.

□

Proof. To state that our equation (3.9) is solvable, it is enough to define a suitable linear, bounded and surjective operator $\hat{b} : E_2^{(+)} \to E_2$ and apply Theorem 3.2. Put, by definition,

$$\hat{b}\ z := y, \tag{3.13}$$

where $z := (y,t) \in E_2^{(+)}$, $y \in E_2$, $t \in \mathbb{R}$. The operator (3.13) is evidently linear bounded with the norm $||\hat{b}|| = 1$ and surjective with $Range\ \hat{b} = E_2$. Take now the mapping $\varphi := \varphi_r^{(\varepsilon)} : E_2^{(+)} \to E_2$

for $\varepsilon \in \mathbb{R}\setminus\{0\}$ and apply Theorem 3.1. Since $\dim Ker\ \hat{b} = 1$, we get that equation (3.9), written in the form

$$\varphi(z) := \varphi_r^{(\varepsilon)}(z) = \hat{b}\ z \tag{3.14}$$

for all $z \in E_2^{(+)}$, possesses a nonempty solution set $\mathcal{N}(\hat{b}, \varphi_r^{(\varepsilon)}) \subset E_2^{(+)}$, whose topological dimension $\dim \mathcal{N}(\hat{b}, \varphi_r^{(\varepsilon)}) \geq 0$ for all $\varepsilon \in \mathbb{R}\setminus\{0\}$. Assume now, for a moment, that the value $t_\varepsilon \neq 0$. Then, based on expression (3.11), one can easily get that the well-defined vector

$$x_\varepsilon := \frac{r t_\varepsilon (s(y_\varepsilon) + t_\varepsilon \bar{c})}{|t_\varepsilon|\ \|s(y_\varepsilon) + t_\varepsilon \bar{c}\|_1} \tag{3.15}$$

satisfies the following equation:

$$f(x_\varepsilon) = t_\varepsilon^{-2}(t_\varepsilon^2 + \varepsilon^2)\hat{a}\ x_\varepsilon. \tag{3.16}$$

Really, from (3.11) we obtain that

$$\begin{aligned} \frac{t_\varepsilon}{t_\varepsilon^2+\varepsilon^2} f_r(t_\varepsilon s(y_\varepsilon) + t_\varepsilon^2 \bar{c}) &= \frac{t_\varepsilon\, |t_\varepsilon|\ \|s(y_\varepsilon) + t_\varepsilon \bar{c}\|_1}{r(t_\varepsilon^2+\varepsilon^2)}\ f\left(\frac{r t_\varepsilon (s(y_\varepsilon) + t_\varepsilon \bar{c})}{|t_\varepsilon|\ \|s(y_\varepsilon) + t_\varepsilon \bar{c}\|_1}\right) \\ &= \frac{t_\varepsilon\, |t_\varepsilon|\ \|s(y_\varepsilon) + t_\varepsilon \bar{c}\|_1}{r(t_\varepsilon^2+\varepsilon^2)} f(x_\varepsilon) = y_\varepsilon. \end{aligned} \tag{3.17}$$

Whence, recalling the identity $\hat{a}(s(y_\varepsilon)) = y_\varepsilon$ for any $y_\varepsilon \in E_2$, we find that

$$\begin{aligned} f(x_\varepsilon) &= \frac{(t_\varepsilon^2+\varepsilon^2) r\ \hat{a}\ (s(y_\varepsilon))}{t_\varepsilon\ \|s(y_\varepsilon) + t_\varepsilon \bar{c}\|_1} = \frac{(t_\varepsilon^2+\varepsilon^2)}{t_\varepsilon^2}\ \hat{a}\ \left(\frac{r s(y_\varepsilon) t_\varepsilon}{|t_\varepsilon|\ \|s(y_\varepsilon) + t_\varepsilon \bar{c}\|_1}\right) \\ &= \frac{(t_\varepsilon^2+\varepsilon^2)}{t_\varepsilon^2}\ \hat{a}\ \left(\frac{t_\varepsilon r (s(y_\varepsilon) + t_\varepsilon \bar{c})}{|t_\varepsilon|\ \|s(y_\varepsilon) + t_\varepsilon \bar{c}\|_1}\right) = \frac{(t_\varepsilon^2+\varepsilon^2)}{t_\varepsilon^2}\ \hat{a}\ x_\varepsilon, \end{aligned} \tag{3.18}$$

where we took into account the linearity of the operator $\hat{a} : E_1 \to E_2$ and the fact that the vector $\bar{c} \in Ker\ \hat{a}$. Thereby, the constructed vector $x_\varepsilon \in E_1$ satisfies for $\varepsilon \in \mathbb{R}\setminus\{0\}$ the equation (3.16). The considerations above hold since we assumed that $t_\varepsilon \neq 0$ for all $\varepsilon \in \mathbb{R}\setminus\{0\}$. To show this is the case, assume the inverse that is $t_\varepsilon = 0$ for some $\varepsilon \in \mathbb{R}\setminus\{0\}$. We then get from (3.11) and condition 2) imposed before on the mapping $f : D(f) \subset E_1 \to E_2$ right away that simultaneously there should be fulfilled the equality $\|y_\varepsilon\|_2 = 0$, contradicting to the condition (3.10). Thus, for all $\varepsilon \in \mathbb{R}\setminus\{0\}$ the value $t_\varepsilon \neq 0$. If to state more accurate estimations, mainly, that the following inequalities

$$1 > \varliminf_{\varepsilon\to 0} |t_\varepsilon|^2 \geq 1 - \alpha_0^2 > 0 \tag{3.19}$$

hold for some positive value $\alpha_0 > 0$, then one can try to calculate the limit:

$$\lim_{n\to\infty} f(x_{\varepsilon_n}) = f(x_0) = \lim_{n\to\infty} \left(t_{\varepsilon_n}^{-2}(t_{\varepsilon_n}^2 + \varepsilon_n^2)\ \hat{a}\ x_{\varepsilon_n}\right) = \hat{a}\ x_0 \tag{3.20}$$

for some subsequence $\varepsilon_n \to 0$ as $n \to \infty$. Here we have assumed that there exists $\lim\limits_{n\to\infty} x_{\varepsilon_n} = x_0$, that is

$$\lim_{n\to\infty} \frac{t_{\varepsilon_n} r (s(y_{\varepsilon_n}) + t_{\varepsilon_n} \bar{c})}{|t_{\varepsilon_n}|\ \|s(y_{\varepsilon_n}) + t_{\varepsilon_n} \bar{c}\|_1} = x_0 \tag{3.21}$$

depending on the chosen before nontrivial vector $\bar{c} \in Ker\ \hat{a}$.

Owing to the $\hat{a}$-compactness of the mapping $f : D(f) \subset E_1 \to E_2$ and the continuity of the operators $\tilde{a}^{-1} : E_2 \to \tilde{E}_1$ and $s : E_2 \to E_1$, for the limit (3.21) to exist it is enough only to

state that there holds inequality (3.19). Really, since owing to relationship (3.10) for all $\varepsilon > 0$ the following condition

$$|t_\varepsilon|^2 + \|y_\varepsilon\|_2^2 = 1 \tag{3.22}$$

holds, the limit (3.21) will exist, if to state equivalently that

$$\lim_{n\to\infty} \|y_{\varepsilon_n}\|_2 \le \alpha_0 < 1. \tag{3.23}$$

To show inequality (3.23), consider expression (3.11) and make the following estimations:

$$\begin{aligned}
\lim_{n\to\infty} \|y_{\varepsilon_n}\|_2 &= \lim_{n\to\infty}\left(\frac{|t_{\varepsilon_n}|}{t_{\varepsilon_n}^2+\varepsilon_n^2}\left\|f_r(t_{\varepsilon_n}s(y_{\varepsilon_n})+t_{\varepsilon_n}^2\bar{c})\right\|_2\right)\\
&\le \lim_{n\to\infty}\left(\frac{|t_{\varepsilon_n}|^2}{(t_{\varepsilon_n}^2+\varepsilon_n^2)}\frac{\|s(y_{\varepsilon_n})+t_{\varepsilon_n}\bar{c}\|_1}{r}f\left(\frac{rt_{\varepsilon_n}(s(y_{\varepsilon_n})+t_{\varepsilon_n}\bar{c})}{|t_{\varepsilon_n}|\,\|s(y_{\varepsilon_n})+t_{\varepsilon_n}\bar{c}\|_1}\right)\right)\\
&\le \lim_{n\to\infty}\|s(y_{\varepsilon_n})+t_{\varepsilon_n}\bar{c}\|_1\,k_f^{-1} \le k_f^{-1}(\lim_{n\to\infty}\|s(y_{\varepsilon_n})\|_1+(1-\lim_{n\to\infty}\|y_{\varepsilon_n}\|_2^2)^{1/2}\,\|\bar{c}\|_1) \qquad (3.24)\\
&\le k_f^{-1}(k_s\lim_{n\to\infty}\|y_{\varepsilon_n}\|_2+[1-\lim_{n\to\infty}\|y_{\varepsilon_n}\|_2^2]^{1/2}\,\|\bar{c}\|_1).
\end{aligned}$$

Thus, we obtain from (3.24) that the value $\alpha_0 := \lim_{n\to\infty}\|y_{\varepsilon_n}\|_2 \in \mathbb{R}_+$ satisfies the following inequalities:

$$0 \le \alpha_0 \le k_f^{-1}(k_s\ \alpha_0\ +\ (1-\alpha_0^2)^{1/2}\,\|\bar{c}\|_1) \le 1 \tag{3.25}$$

where, in general, $\alpha_0 \in [0,1]$. For inequalities (3.25) to hold true, we need to consider two possibilities:

$$a)\ \ k_s k_f^{-1} \ge 1\ ; \qquad b)\ k_s k_f^{-1} < 1. \tag{3.26}$$

For the case $a)$ of (3.26) we can easily state that

$$1 \le \min(\frac{k_s}{k_f},1) \le \alpha_0 \le k_f^{-1}\sqrt{k_s^2+\|\bar{c}\|_1^2}\ . \tag{3.27}$$

For the case $b)$ of (3.27) one gets similarly that

$$0 \le \alpha_0 \le \frac{\|\bar{c}\|_1}{\sqrt{\ \|\bar{c}\|_1^2+(k_s-k_f)^2}}\ . \tag{3.28}$$

Since we are interested in any value of $\alpha_0 < 1$, the only inequality (3.28) fits to the searched for exact inequality

$$0 \le \alpha_0 \le \frac{\|\bar{c}\|_1}{\sqrt{\ \|\bar{c}\|_1^2+(k_s-k_f)^2}} < 1, \tag{3.29}$$

guaranteeing the existence of a nontrivial (not zero!) solution to equation (3.20). Thereby, the nontrivial vector $x_0 \in D(f)$ constructed above satisfies, following from (3.20), the equality

$$f(x_0) = \hat{a}\ x_0. \tag{3.30}$$

Moreover, since the vector $x_0 \in D(f)$, owing to representation (3.21), depends nontrivially on the chosen vector $\bar{c} \in Ker\ \hat{a}$, we deduce that the corresponding to (3.30) solution set $\mathcal{N}(\hat{a},f) \subset E_1$ is nonempty, if $\dim Ker\ \hat{a} \ge 1$, and the topological dimension $\dim \mathcal{N}(\hat{a},f) \ge \dim Ker\ \hat{a}\ -1$. The latter finishes the proof of the theorem. □

4. Corollaries

The classical Leray-Schauder fixed point theorem, as is well known [1, 2, 13, 15, 18], reads as follows.

Theorem 4.1. *Let a compact mapping $\bar{f}: B \to B$ in a Banach space B is such that there exists a closed convex and bounded set $M \subset B$, for which $\bar{f}(M) \subseteq M$. Then there exists a fixed point $\bar{x} \in M$, such that*

$$\bar{f}(\bar{x}) = \bar{x}. \tag{4.1}$$

Proof. One can present two completely different approaches to the proof of this classical Leray-Schauder theorem, using the main Theorem 3.1. The first one is based on simple geometrical considerations, and the second one, requires some topological backgrounds. □

Proof. **Approach 1.** Put, by definition, that $E_1 := B \oplus \mathbb{R}$, $E_2 := B$ and $M_f := Conv$ $\bar{f}(M) \subseteq M$ is the convex and compact convex hull of the image $\bar{f}(M) \subseteq M$. For any point $x \in B$ one can define the set-valued projection mapping

$$B \ni x \to P_{M_f}(x) \subset M_f \subset B, \tag{4.2}$$

where

$$\inf_{y \in M_f} ||x - y|| := ||x - P_{M_f}(x)||. \tag{4.3}$$

The set-valued mapping (4.2) is well defined and upper semi-continuous [3, 4] owing to the closedness, boundedness and convexity of the set $M_f \subset B$. Now take the unit sphere $S_1(0) \subset E_1$ and construct a mapping $f: S_1(0) \subset E_1 \to E_2$, where, by definition, for any $(x, \tau) \in S_1(0)$

$$f(x, \tau) := \bar{f}(\bar{P}_{M_f}(x)) - \bar{P}_{M_f}(x) + \hat{b}\ x, \tag{4.4}$$

$\bar{P}_{M_f}: B \to M_f \subset B$ is a suitable continuous selection [14] for the mapping (4.2) and $\hat{b}: B \to B$ is an arbitrary compact and surjective mapping. Concerning the corresponding mapping $\hat{a}:$ $E_1 \to E_2$, we put, by definition,

$$\hat{a}\ (x, \tau) := \hat{b}\ x \tag{4.5}$$

for all $(x, \tau) \in E_1 = B \oplus \mathbb{R}$. It is now easy to observe that the following lemma holds.

Lemma 4.2. *The mapping $f: S_1(0) \subset E_1 \to E_2$, defined by (4.4), is continuous and $\hat{a}-$compact.*

Proof. Really, for any $x \in B$ the element $\bar{P}_{M_f}(x) \in M_f$ and $\bar{f}(\bar{P}_{M_f}(x)) \in M_f$, owing to the invariance $\bar{f}(M) \subseteq M$. From the compactness of the mappings $\bar{f}: M \to M$ and $\hat{b}: B \to B$ one easily gets the $\hat{a}$-compactness of the constructed mapping $f: E_1 \to E_2$ that proves the lemma. □

Now taking into account Lemma 4.2 and the fact that operator $\hat{a}: E_1 \to E_2$, defined by (4.5), is closed and surjective, owing to the assumptions done above, we can apply to the equation

$$\hat{a}\ (x, \tau) = f(x, \tau), \tag{4.6}$$

where $(x,\tau) \in S_1(0) \subset E_1$, the main Theorem 3.1 and, thereby, state that the corresponding solution set $\mathcal{N}(\hat{a}, f) \subset E_1$ is nonempty, since $\dim Ker\ \hat{a} \geq 1$. In particular, from (4.6) one gets that

$$\bar{f}(\bar{P}_{M_f}(x_\tau)) = \bar{P}_{M_f}(x_\tau) \tag{4.7}$$

for the vector $\bar{P}_{M_f}(x_\tau) \in M_f$, where a point $x_\tau \in B_1(0)$ satisfies the condition $||x_\tau||^2 + ||\tau||^2 = 1$ for some value $|\tau| \leq 1$.

Thereby, we have stated that the fixed point problem (4.1) is solvable and its solution can, in particular, be obtained as the projection $\bar{x} := \bar{P}_{M_f}(x_\tau)$ of some point $x_\tau \in B_1(0)$ upon the compact, convex and invariant set $M_f \subseteq M \subset B$.

Approach 2. We shall start from the following result [16, 7] about the general structure of compact and convex sets in metrizible locally convex topological vector spaces, being a weak version of the well known Krein-Milman theorem.

Lemma 4.3. *Let E be a metrizible locally convex topological vector space over the fileld $\mathbb{R}$, $F \subset E$ be its dense vector subspace and $M \subset E$ be any convex and closed compact subset. Then there exists a countable linearly independent sequence $\{e_n \in F : n \in \mathbb{Z}_+\}$, such that $\lim_{n\to\infty} e_n \to 0$, a countable sequence $\{\lambda_n(x) \in \mathbb{R} : n \in \mathbb{Z}_+\}$, such that*

$$\sum_{n\in\mathbb{Z}_+} |\lambda_n(x)| \leq 1, \tag{4.8}$$

and every element $x \in M$ allows the representation

$$x = \sum_{n\in\mathbb{Z}_+} \lambda_n(x) e_n. \tag{4.9}$$

Proof. A proof of this lemma can be found, for instance, in [16, 7], so we will not present it here. □

As any Banach space B is a metrizible locally convex topological vector space, representation (4.9) naturally generates a well-defined surjective and continuous compact mapping $\xi : l_1(\mathbb{Z}_+;\mathbb{R}) \to M_f \subset B$ with the domain $D(\xi) = \bar{B}_1(0)$, where the set $\bar{B}_1(0) \subset l_1(\mathbb{Z}_+;\mathbb{R})$ is the unit ball centered at zero in the Banach space $l_1(\mathbb{Z}_+;\mathbb{R})$ and $M_f := Conv\ \bar{f}(M) \subseteq M$ is, as before, the convex and compact convex hull of the image $\bar{f}(M) \subseteq M$. The next lemma follows from Lemma 4.3 and [16, 7] and some related results about the continuous selections from [8, 2, 12, 18].

Lemma 4.4. *There exists such a continuous selection $\xi_s^{-1} : B \supset M_f \to \bar{B}_1(0) \subset l_1(\mathbb{Z}_+;\mathbb{R})$, $\xi \cdot \xi_s^{-1} = id : M_f \to M_f$, that for any vector $x \in M_f$ the value $\xi_s^{-1}(x) \in \bar{B}_1(0)$ determines uniquely this vector by means of representation (4.9) as*

$$x = \sum_{n\in\mathbb{Z}_+} (\xi_s^{-1}(x))_n e_n. \tag{4.10}$$

Moreover, this selection can be chosen in such a way, that an induced mapping $\bar{F}_s : l_1(\mathbb{Z}_+;\mathbb{R}) \supset \bar{B}_1(0) \to \bar{B}_1(0) \subset l_1(\mathbb{Z}_+;\mathbb{R})$, *defined as*

$$\bar{F}_s(\lambda) := \xi_s^{-1} \cdot \bar{f}(\xi(\lambda)) \tag{4.11}$$

for any $\lambda \in \bar{B}_1(0) \subset l_1(\mathbb{Z}_+;\mathbb{R})$, *is continuous and also compact.*

Proof. Modulo the existence [14, 3] of a selection $\xi_s^{-1} : B \supset M_f \to \bar{B}_1(0) \subset l_1(\mathbb{Z}_+;\mathbb{R})$, a proof is based both on representations (4.10) and (4.11) and on the compactness of the mapping $\xi : l_1(\mathbb{Z}_+;\mathbb{R}) \supset \bar{B}_1(0) \to M_f \subset B$ and the set M_f , as well as on the standard fact [13, 18] that the continuous image of a compact set is compact too. □

Pose now the fixed point problem for the compact mapping $\bar{F}_s : l_1(\mathbb{Z}_+;\mathbb{R}) \supset \bar{B}_1(0) \to \bar{B}_1(0) \subset l_1(\mathbb{Z}_+;\mathbb{R})$ constructed above as follows:

$$\bar{F}_s(\bar{\lambda}) := \bar{\lambda} \tag{4.12}$$

for some point $\bar{\lambda} \in \bar{B}_1(0)$.

The solution of the fixed point equation (4.12) is, evidently, completely equivalent to proving Theorem 4.1, since the corresponding vector $\bar{x} := \xi(\bar{\lambda}) \in M_f$, owing to definition (4.11), satisfies the following relationships:

$$\bar{f}(\bar{x}) = \bar{f}(\xi(\bar{\lambda})) = \xi(\bar{F}_s(\lambda)) \Rightarrow \xi(\bar{\lambda}) = \bar{x}. \tag{4.13}$$

Thereby, the vector $\bar{x} := \xi(\bar{\lambda}) \in M_f$ solves fixed the point problem (4.1) for the compact mapping $\bar{f} : B \to B$.

To prove the existence of a solution to equation (4.12), we will construct the suitable Banach spaces $E_1 := l_1(\mathbb{Z}_+;\mathbb{R}) \oplus \mathbb{R}$ and $E_2 := l_1(\mathbb{Z}_+;\mathbb{R})$ and take the unit sphere $S_1(0) \subset E_1$, consisting of points $(\lambda, \tau) \in E_1$, for which $||\lambda|| + |\tau| = 1$. The mapping $\bar{F}_s : \bar{B}_1(0) \to \bar{B}_1(0)$, constructed above, one can extend upon the sphere $S_1(0) \subset E_1$, defining a mapping $f : E_1 \supset S_1(0) \to \bar{S}_1(0) \subset E_2$ as

$$f(\lambda, \tau) := \bar{F}_s(\lambda) \tag{4.14}$$

for any $(\lambda, \tau) \in S_1(0) \subset E_1$. A suitable linear, closed and surjective operator $\hat{a} : E_1 \to E_2$ one can define as

$$\hat{a}\ (\lambda, \tau) := \lambda \tag{4.15}$$

for all $(\lambda, \tau) \in E_1$. The resulting equation

$$\hat{a}\ (\lambda, \tau) = f(\lambda, \tau) \tag{4.16}$$

for $(\lambda, \tau) \in S_1(0) \subset E_1$ exactly fits into the conditions formulated in Theorem 3.1, being simultaneously equivalent to fixed point problem (4.12) for the mapping $\bar{F}_s : \bar{B}_1(0) \to \bar{B}_1(0)$. Since $\dim Ker\ \hat{a} = 1$, there exists the nonempty solution set $\mathrm{N}(\hat{a}, f) \subset E_1$ of equation (4.16). If

a point $(\lambda_\tau, \tau) \in \mathrm{N}(\hat{a}, f) \subset S_1(0)$, where $||\lambda_\tau|| + |\tau| = 1$ for some value $|\tau| \leq 1$, then the fixed point equality

$$\bar{F}_s(\ \lambda_\tau) := \lambda_\tau \tag{4.17}$$

holds for the value $\lambda_\tau \in \bar{B}_1(0) \subset l_1(\mathbb{Z}_+; \mathbb{R})$. Having denoted now $\lambda_\tau :=\ \bar{\lambda} \in \bar{B}_1(0)$, we get, owing to relationships (4.13), the corresponding solution to the fixed point problem for the compact mapping $\bar{f} : B \to B$, thereby finishing the proof of the Leray-Schauder theorem 4.1. □

There exist, evidently, many other interesting applications of the main Theorem 3.1 in particular, proving the existence theorem for diverse types of differential equations in Banach spaces with both fixed boundary conditions and inclusions [1, 2, 8, 11, 10, 15]. These and related research problems we plan to study in move detail in another paper.

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